The Wisdom of Crowds: Voting and Information Aggregation

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Crowds Make Choices

Decisions of a large number of people may be informative:

- Stock markets
- Sports betting
- Elections
- Opinion polls
- Jury trials
- Popularity of restaurants, books and movies

Madness or Wisdom of Crowds?

“No one in this world, so far as I know, has ever lost money by underestimating the intelligence of the great masses of the common people.”
H. I. Mencken.

“If the blind lead the blind, both shall fall into the ditch.”
Matthew 15:14.

“A large group of diverse individuals will come up with better and more robust forecasts and make more intelligent decisions than even the most skilled decision maker.”
James Surowiecki.

Wisdom of Crowds: Examples

- Francis Galton and the Plymouth ox weighing competition
- Explosion of space shuttle *Challenger* and the stock price of Morton Thiokol
- *Who Wants to be a Millionaire*? Accuracy rate of audience (91%) better than that of expert friend (65%)
- Iowa electronic market predicts election results:
  - Vote share predictions within 1.37% in US presidential elections, 3.43% in other
  - Better than 75% of opinion polls
- Hollywood Stock Exchange predicts Oscar winners

Voting and Mechanism Design

- Voting serves two purposes:
  - aggregate preferences and resolve conflict of interest
  - aggregate information on common interest aspects of choice
- Example: disagreements on free trade can spring from
  - the fact that it produces winners and losers
  - no one is sure about macroeconomic implications
- We focus on the "pure" information aggregation problem
- Restricted mechanism design: no communication

Condorcet Jury Theorem

1. The decision of a jury will be correct more often than the decision of any single individual (Generalization: larger juries do better than smaller ones).
2. The decision of a jury is correct with probability approaching 1 a the size of the jury grows to infinity.

- Under what conditions do these conclusions hold?
- Which voting voting rules satisfy these properties?
- Statistical versus strategic jury theorems.

Voting Over a Binary Choice

- State of the world (s) = guilty (G) or innocent (I).
- Decision (d) = convict (C) or acquit (A).
- Correct decision: C when G, A when I.
- Voters $1, 2, 3..., n$. Probability of j voting correctly = $p_j \in [\frac{1}{2}, 1]$. Probabilities are independent.
- Voting rule = $\alpha \in [\frac{1}{2}, 1]$ (minimum fraction of votes needed for a decision).
- Let $x_j = 1$ when j's vote is correct; $x_j = 0$ when wrong.
- Probability that the jury's decision is correct:

$$P(n, \alpha) = \Pr\left[X = \sum_{j=1}^{n} x_j \geq \alpha n\right]$$

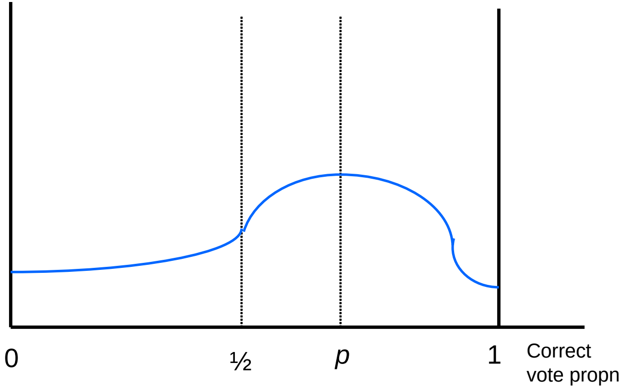
Statistical Jury Theorem

Theorem

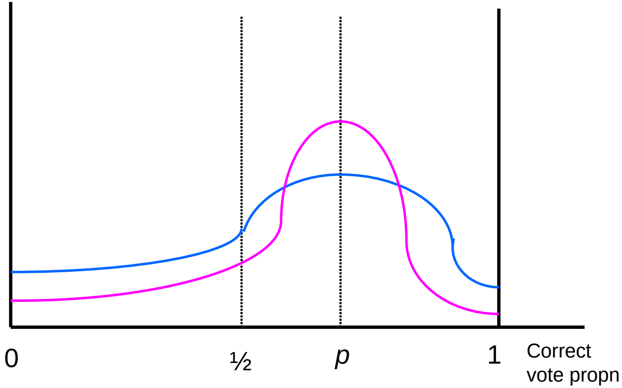
Assume $p_j = p$ for all j. Then
(1) If $p > \alpha$, then there exists N such that for all $n > N$, $P(n, \alpha) > p$ and $\lim_{n\to\infty} P(n, \alpha) = 1$.
(2) If $p \leq \alpha$, then there exists N such that for all $n > N$, $P(n, \alpha) < p$ and $\lim_{n\to\infty} P(n, \alpha) = 0$.

- Under majority rule, the jury theorems hold.
- Under super-majority rule, individual voters must be sufficiently accurate for the theorems to be valid.
- The *ex ante* probability of a decision (e.g., conviction) or an error (e.g., convicting the innocent) is increasing in n.

Increasing Jury Size



Increasing Jury Size



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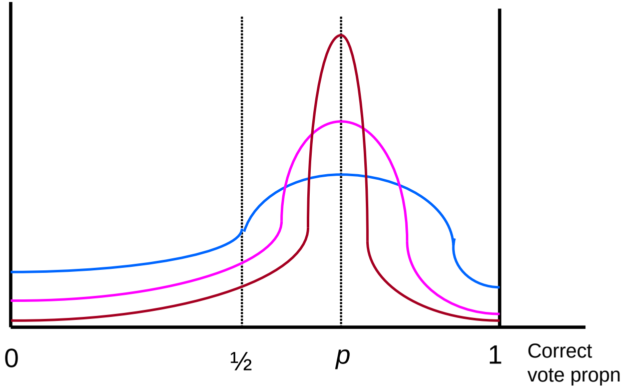

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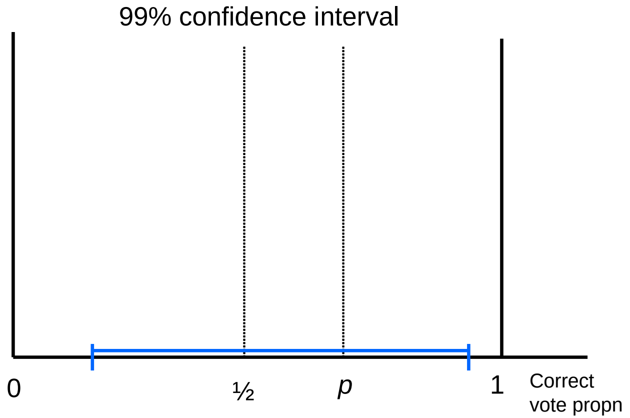
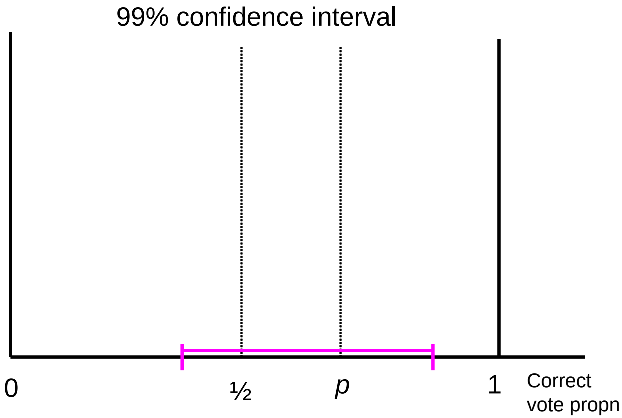


Increasing Jury Size



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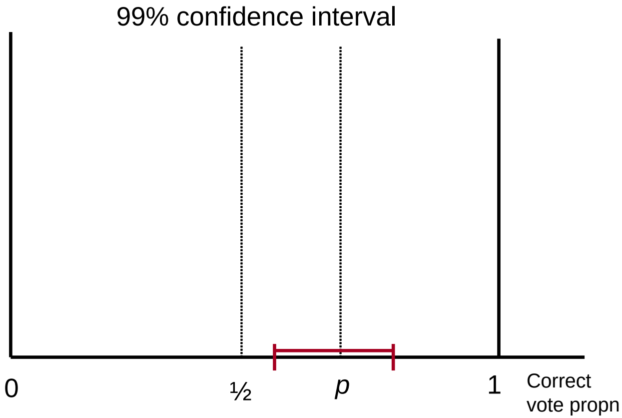

A Model of Rational Voters

- Voter preferences:

$$u(s,d) = \begin{cases} -q & \text{if } s = I, d = C, \\ -(1-q) & \text{if } s = G, d = A, \\ 0 & \text{otherwise,} \end{cases}$$

where $q \in (0,1)$.

- q is the "threshold of doubt": C is optimal iff the voter believes there is a greater than q chance the state is G.
- Let prior on "guilt" $= \pi$.

A Model of Rational Voters

- Voter private information: conditionally independent private signal $t_j \in \{g, i\}$, with distribution

	g	i
G	p_G	$1 - p_G$
I	$1 - p_I$	p_I

- Signals are informative: $p_G \neq 1 - p_I$.
- Voters **cannot communicate**; they must vote independently.
- Since there is common interest, the game with communication is trivial: voters have the incentive to share their signals truthfully.

Strategies

- Mappings from signal to vote: $\sigma : \{g, i\} \to \Delta\{C, A\}$.
- A strategy is **informative** if $\sigma(g) = 1$ and $\sigma(i) = 0$.
- A strategy is **responsive** if $\sigma(g) \neq \sigma(i)$.
- A strategy is **sincere** if it is the same way the juror would have voted if she were making the decision alone ($n = 1$).

An Example

- Two voters or one
- Threshold of doubt: $q = \frac{1}{2}$
- Prior on $G = \pi = \frac{2}{3}$
- Signal accuracy: $p = \frac{3}{4}$

A Judge $(n = 1)$

- $\Pr(G|g) = \frac{\pi p}{\pi p + (1-\pi)(1-p)} = \frac{6}{7} > \frac{1}{2}$
- $\Pr(G|i) = \frac{\pi(1-p)}{\pi(1-p)+(1-\pi)p} = \frac{2}{5} < \frac{1}{2}$
- Optimal decision is informative: $\sigma(g) = 1$ and $\sigma(i) = 0$
- Expected payoff $= -\frac{1}{2}.\frac{1}{4} = -0.125$

Jury of Two

- Assume unanimity required for C, otherwise A
- Expected payoff:

$$-\frac{1}{2}[\pi(1-p^2)+(1-\pi)(1-p)^2] = -0.15625$$

- But sincere voting is not a Nash equilibrium
- Assume sincere voting. Then

$$\Pr(G|piv, i) = \Pr(G|g, i) = \frac{2}{3} > \frac{1}{2}$$

- Voter receiving i signal will want to deviate and vote for C

Symmetric Mixed Equilibrium

- Let $\sigma(g) = 1$ and $\sigma(i) = \sigma$

$$\begin{aligned} \Pr(G|piv) &= \frac{\pi\left[p + (1-p)\sigma\right]}{\pi\left[p + (1-p)\sigma\right] + (1-\pi)\left[p\sigma + 1 - p\right]} \\ &= \frac{6+2\sigma}{7+5\sigma} = \lambda \end{aligned}$$

$$\begin{aligned} \Pr(G|piv, i) &= \frac{\lambda(1-p)}{\lambda(1-p) + (1-\lambda)p} \\ &= \frac{\lambda}{3-2\lambda} \end{aligned}$$

- Indifference $\Rightarrow \frac{\lambda}{3-2\lambda} = \frac{1}{2} \Rightarrow \lambda = \frac{3}{4} \Rightarrow \sigma = \frac{3}{7}$

Symmetric Mixed Equilibrium

- Error probabilities:
  - In state G: $1 - \left[p^2 + 2p(1-p)\sigma + (1-p)^2\sigma^2\right] = \frac{13}{49}$
  - In state I: $(1-p)^2 + 2p(1-p)\sigma + p^2\sigma^2 = \frac{16}{49}$
- Expected payoff $= -\frac{1}{2}.\left[\frac{2}{3}.\frac{13}{49} + \frac{1}{3}.\frac{16}{49}\right] = -0.143 < 0.125$
- Jury still does worse than judge, even with sophisticated voters
- Is this adverse consequence a result of equilibrium selection?
- Another equilibrium: both voters vote for A regardless of their signal. Since neither is pivotal, best response property is not violated!
- Is there a better equilibrium than all of these?

Asymmetric Pure Equilibrium

- Voter 1 votes for C regardless of signal
- Voter 2 votes sincerely
- Since voter 2 is always pivotal, he is effectively a judge. Hence sincere and informative voting is a best response for voter 2.
- Checking best response property for voter 1:

$$\Pr(G|piv, i) = \Pr(G|g, i) = \pi = \frac{2}{3} > \frac{1}{2}$$

- The equilibrium mimics trial by judge ($n = 1$)
- Expected payoff $= -0.125$
- Jury does no worse than judge under this equilibrium selection.

Full Information and Sincere Voting

- If all the signals were known, posterior belief:

$$\Pr[s = G | \#g \text{ signals is } k] = \frac{\pi p_G^k (1 - p_G)^{n-k}}{\pi p_G^k (1 - p_G)^{n-k} + (1 - \pi)(1 - p_I)^k p}$$

$$= \frac{1}{1 + \frac{1-\pi}{\pi} \left[\frac{1-p_I}{p_G} \right]^k \left[\frac{p_I}{1-p_G} \right]^{n-k}}$$

- There is a critical number of g signals, k^*, such that the posterior is λ or higher iff $k \geq k^*$.

Theorem

If $p_I = p_G = p$, sincere voting is informative and rational iff the minimum number of votes needed for conviction is exactly k^*.

Equilibrium Selection Issues

Theorem

(McLennan, 1998) Generically, in any common interest game, the efficient quilibrium is in mixed strategies.

- Symmetric mixed equilibria are generally not Pareto efficient in the class of equilibria.
- Efficient equilibria are asymmetric:
  - a subset of voters vote uninformatively, stacking votes for C or A regardless of their signal
  - remaining voters vote informatively, i.e., $\sigma(g) = 1$, $\sigma(i) = 0$
- The most efficient voting rule is one where the set of uninformative voters is zero.

Inferiority of Unanimous Verdicts (Feddersen and Pesendorfer, 1997)

- Under sincere voting, raising the minimum votes needed for conviction lowers the probability of wrongful conviction.
- Under strategic voting, *both* error probabilities may go up.
- Relies on the information content of being "pivotal".
- Unanimity
  - makes conviction harder for fixed voting behavior.
  - makes voters more willing to convict.
- CJT fails for unanimity rule but not for interior rules.
- Generalized in Chakraborty and Ghosh (2003).

Symmetric Mixed Equilibria

- Let $\pi = \frac{1}{2}$ and $p_G = p_I = p$.
- Let $\sigma(g), \sigma(i)$ be probability of voting for C when signal is and, i respectively.
- An equilibrium is responsive if $\sigma^*(g) \neq \sigma^*(i)$.
- Probabilities of voting for C:

$$\begin{aligned} \gamma_G &= p\sigma(g) + (1-p)\sigma(i) \\ \gamma_I &= (1-p)\sigma(g) + p\sigma(i) \end{aligned}$$

- Since posterior after a g signal $>$ posterior after an i signal,

$$\begin{aligned} \sigma(g) &\in (0,1) \Rightarrow \sigma(i) = 0 \\ \sigma(i) &\in (0,1) \Rightarrow \sigma(g) = 1 \end{aligned}$$

Mixed Equilibrium: Type 1

- $\sigma^*(g) = 1$ and $\sigma^*(i) = 0$.
- Arises if

$$\underbrace{\frac{p^{\alpha n-1}(1-p)^{(1-\alpha)n+1}}{p^{\alpha n-1}(1-p)^{(1-\alpha)n+1} + (1-p)^{\alpha n-1}p^{(1-\alpha)n+1}}}_{\Pr(G|piv,i)} \quad \begin{array}{l} \leq \quad q \\ \leq \quad \text{doubt threshold} \end{array}$$

- and

$$\underbrace{\frac{p^{\alpha n}(1-p)^{(1-\alpha)n}}{p^{\alpha n}(1-p)^{(1-\alpha)n} + (1-p)^{\alpha n}p^{(1-\alpha)n}}}_{\Pr(G|piv,g)} \quad \begin{array}{l} \geq \quad q \\ \geq \quad \text{doubt threshold} \end{array}$$

Mixed Equilibrium: Type 2

- $\sigma^*(g) = 1$ and $\sigma^*(i) = \sigma$.
- Indifference after signal i implies:

$$\frac{(1-p)\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n}}{(1-p)\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n} + p\gamma_I^{\alpha n-1}(1-\gamma_I)^{(1-\alpha)n}} = q$$

- Use

$$\gamma_G = p + (1-p)\sigma; \;\; \gamma_I = p\sigma + (1-p)$$

- On solving:

$$\begin{aligned} \sigma(i) &= \frac{p(1+f)-1}{p-f(1-p)} \\ \text{where } f &= \left(\frac{1-q}{q}\right)^{\frac{1}{\alpha n-1}} \left(\frac{1-p}{p}\right)^{\frac{(1-\alpha)n+1}{\alpha n-1}} \end{aligned}$$

Mixed Equilibrium: Type 3

- $\sigma^*(g) = \sigma$ and $\sigma^*(i) = 0$.
- Indifference after signal g implies:

$$\frac{p\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n}}{p\gamma_G^{\alpha n-1}(1-\gamma_G)^{(1-\alpha)n} + (1-p)\gamma_I^{\alpha n-1}(1-\gamma_I)^{(1-\alpha)n}} = q$$

- Use

$$\gamma_G = p + (1-p)\sigma; \;\; \gamma_I = p\sigma + (1-p)$$

- On solving:

$$\begin{aligned} \sigma(g) &= \frac{h-1}{p(h+1)-1} \\ \text{where} \;\; h &= \left(\frac{1-q}{q}\right)^{\frac{1}{(1-\alpha)n}} \left(\frac{1-p}{p}\right)^{\frac{\alpha n}{(1-\alpha)n}} \end{aligned}$$

Interior Rules

Theorem

Suppose $\alpha \in (0,1)$. (1) There is $\overline{n}$ such that for all $n \geq \overline{n}$, there is a symmetric responsive equilibrium. (2) For symmetric, responsive equilibria

$$\lim_{n \longrightarrow \infty} \Pr(C|I) = \lim_{n \longrightarrow \infty} \Pr(A|G) = 0$$

- Both error probabilities (convicting the innocent and acquitting the guilty) vanish as the size of the jury becomes very large.
- Note that *any* interior rule has this property.

Interior Rules

- Limit expressions for mixtures:

$$\lim_{n\to\infty} \sigma(i) = \frac{p\left[1+\left(\frac{1-p}{p}\right)^{\frac{1-\alpha}{\alpha}}\right]-1}{p-\left(\frac{1-p}{p}\right)^{\frac{1-\alpha}{\alpha}}(1-p)} \in (0,1)$$

$$\lim_{n\to\infty} \sigma(g) = \frac{\left(\frac{1-p}{p}\right)^{\frac{\alpha}{1-\alpha}}-1}{p\left[\left(\frac{1-p}{p}\right)^{\frac{\alpha}{1-\alpha}}+1\right]-1} \in (0,1)$$

- East to check the theorem holds.

Outline of Proof

- As $n \longrightarrow \infty$, the following holds:

$$\gamma_I < \alpha < \gamma_G$$

- By the Law of Large Numbers, for large n, the proportion of votes for C is γ_G (when guilty) and γ_I (when innocent).
- Hence, almost surely, the decision is C (when guilty) and A (when innocent).
- For most voting rules, Condorcet's conclusions are valid.

Unanimity Rule

Theorem

Under unanimity rule, if the defendant is convicted with strictly positive probability, then $\Pr(I|C)$ *is bounded below by*

$$\min\left\{\frac{1}{2}, \frac{(1-q)(1-p)^2}{(1-p)^2+q(2p-1)}\right\}$$

Unanimity Rule

Theorem

Assume condition 1 and $q > 1 - p$*. Under unanimity rule, there is a unique responsive symmetric equilibrium with the limiting properties:*

$$\begin{aligned} \lim_{n\to\infty} \sigma(i) &= 1 \\ \lim_{n\to\infty} \Pr(C|I) &> 0 \\ \lim_{n\to\infty} \Pr(A|G) &> 0 \end{aligned}$$

An Example

- $p = 0.8, q = 0.9, n = 12.$

Voting rule	7	8	9	10	11	12
$\Pr(C\|I)$	.004	.0011	.0025	.0045	.0066	.0069
$\Pr(A\|G)$	.019	.066	.135	.245	.420	.654
$\sigma(i)$	0	.023	.143	.277	.423	.575

- From $k = 8$ onwards, both error probabilities go up as threshold of conviction is raised. Inefficient regardless of preference parameter q.

Selecting Efficient not Symmetric Equilibria (Chakraborty and Ghosh, 2003)

- Monotonic voting rules: let $k(n)$ be the minimum number of votes necessary for conviction in a jury of size n, and let $k(n)$ as well as $n - k(n)$ be non-decreasing.
- Let $V(n, k(n))$ be jurors' expected payoff if the most efficient equilibrium is played.

Theorem

(a) $V(n, k(n))$ *is non-decreasing in if* $k(n)$ *is a monotonic voting rule (b)* $\lim_{n\to\infty} V(n, k(n)) = 0$ *if and only if* $\lim_{n\to\infty} k(n) = \lim_{n\to\infty}[n - k(n)] = \infty$.

- Larger juries can always mimic the results of smaller juries.
- If required votes are bounded, set of informative voters bounded.